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EIGHT DIMENSIONAL NONALTERNATIVE ALGEBRAS

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Abstract. The purpose of this article is to construct unital 8 dimensional, nonalternative, nonflexible hypercomplex $\mathbb{Z}_2$-graded algebras unlike the octonions by means of the unital 4 dimensional, commutative, nonassociative hypercomplex number system H^*. We also establish some algebraic properties of those algebras.

1. Introduction

As is well known, the extension of number systems $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$ doesn't guarantee the commutative property and associative property. The quaternions $\mathbb{H}$ is a unital 4 dimensional noncommutative, associative, normed division algebra and the octonions $\mathbb{O}$ was constructed by means of Cayley-Dickson construction using quaternions $\mathbb{H}$. Note that $\mathbb{O}$ is a unital 8 dimensional, alternative, noncommutative, normed division algebra and many properties and mathematical applications of the quaternions and the octonions have been explored [1,4,6,9,10].

In 1892, Segre introduced commutative quaternions [5] and interesting properties of commutative quaternions have been established [2,3]. In [7], we introduced extended commutative number systems. Also, we constructed modified 4 dimensional commutative quaternions H^* that is not associative [8].

Recall the 4 dimensional, nonassociative, nonalternative, commutative $\mathbb{Z}_2$-graded algebra H^*.

$$H^* = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_0, a_1, a_2, a_3 \in \mathbb{R},\ i, j, k \notin \mathbb{R}\},$$

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where

$$i^2 = k^2 = -1,\ j^2 = 1,\ ij = ji = k,\ jk = kj = -i,\ ki = ik = j.$$

In [8], we constructed a unital 8 dimensional, nonalternative, noncommutative hypercomplex number system $H_8^* = \{(\alpha, \beta) | \alpha, \beta \in H^*\}$. The product of elements in H_8^* is given by $(\alpha, \beta)(\gamma, \eta) = (\alpha\gamma - \beta\eta^{(1)}, \alpha\eta + \beta\gamma^{(1)})$, where $\alpha^{(1)} = a_0 - a_1 i - a_2 j + a_3 k$ for an element $\alpha = a_0 + a_1 i + a_2 j + a_3 k$.

Throughout this paper, we will denote the algebra H_8^* by $H_8^{*(1)}$. Also, we will let $K = \{(\alpha, \beta) | \alpha, \beta \in H^*\}$ be just a set without considering any operation for convenience.

As we shall see, the newly constructed algebra $H_8^{*(m)},\ m = 2, 3$ will be introduced and some intriguing algebraic properties will be explored in section 2.

Also, some algebraic properties of $M_{n\times n}(H_8^{*(m)})$ will be investigated in section 3.

2. Properties of Eight Dimensional Nonalternative, Noncommutative Algebra

In [9], the conjugate $\alpha^{(1)}$ of $\alpha \in H^*$ was defined. Two more conjugates will be defined as follows:

Definition 2.1. Let $\alpha = a_0 + a_1 i + a_2 j + a_3 k \in H^*$. Then, we define m-th conjugate of α for $m = 2, 3$ as follows:

$$\alpha^{(2)} = a_0 - a_1 i + a_2 j - a_3 k, \qquad \alpha^{(3)} = a_0 + a_1 i - a_2 j - a_3 k.$$

Note that $\alpha = (a_0 + a_1 i) + (a_3 + a_2 i)k$ and if $\alpha_1 = a_0 + a_1 i$ and $\alpha_2 = a_3 + a_2 i$, then

$$\alpha^{(2)} = \overline{\alpha_1} - \overline{\alpha_2} k, \qquad \alpha^{(3)} = \alpha_1 - \alpha_2 k.$$

Here, $\overline{\alpha_t}$ is the complex conjugate of α_t for $t = 1, 2$.

The next properties are obvious.

Theorem 2.2. *Let $\alpha, \beta \in H^*$. Then,*

(1) $\alpha^{(m)} = \alpha$ for all $m = 2, 3$ if and only if $\alpha \in \mathbb{R}$.

(2) $(\alpha^{(m)})^{(m)} = \alpha$ for all $m = 2, 3$.

(3) $(\alpha + \beta)^{(m)} = \alpha^{(m)} + \beta^{(m)}$ for all $m = 2, 3$.

(4) $(\alpha\beta)^{(m)} = \alpha^{(m)}\beta^{(m)}$ for all $m = 2, 3$.

(5) If $\alpha = a_0 + a_1 i + a_2 j + a_3 k$, then

$$\left\{(\alpha\alpha^{(1)})\alpha^{(2)}\right\}\alpha^{(3)} = (a_0^2 + a_1^2 - a_2^2 - a_3^2)^2 + 4(a_0a_3 - a_1a_2)^2.$$

THEOREM 2.3. *Let $f : H^* \longrightarrow \mathbb{R}$ be the function defined by $f(\alpha)^4 = \left\{(\alpha\alpha^{(1)})\alpha^{(2)}\right\}\alpha^{(3)}$ for all $\alpha \in H^*$. Then f satisfies the following properties:*
(1) $f(a\alpha) = |a|f(\alpha)$ for all $a \in \mathbb{R}$.
(2) $f(\alpha + \beta) \leq f(\alpha) + f(\beta)$ is not true in general.

Proof. (1) is obvious. For (2), let $\alpha = 1 + i$ and $\beta = 1 + j$. Then, α and β don't satisfy the inequality. □

Now, we begin by defining the modified algebra $H_8^{*(2)}$ as follows:

DEFINITION 2.4. Let $H_8^{*(2)}$ be the algebra defined by the operations on the set K as follows:

$$(\alpha, \beta) + (\gamma, \eta) = (\alpha + \gamma, \beta + \eta)$$
$$a(\alpha, \beta) = (a\alpha, a\beta)$$
$$(\alpha, \beta)(\gamma, \eta) = (\alpha\gamma - \beta\eta^{(2)}, \alpha\eta + \beta\gamma^{(2)})$$

for all elements $(\alpha, \beta), (\gamma, \eta) \in K$ and $a \in \mathbb{R}$.

As we shall see, $H_8^{*(2)}$ loses commutative property and flexible property as follows:

THEOREM 2.5. *$H_8^{*(2)}$ is an 8 dimensional nonalternative, noncommutative, nonflexible, nondivision algebra with the identy.*

Proof. Firstly, noncommutative property is obvious. Note that

$$\{(j, k)(j, k)\}(i, j) = (2k, 2) \neq (-2i, 0) = (j, k)\{(j, k)(i, j)\}$$

Thus, $H_8^{*(2)}$ is a nonalternative algebra.

Moreover, $H_8^{*(2)}$ is a nonflexible algebra since

$$\{(i, j)(j, i)\}(i, j) = 2(j, -i) \neq 2(j, i) = (i, j)\{(j, i)(i, j)\}.$$

To show $H_8^{*(2)}$ is not a division algebra, consider the elements $(1+i, 0)$ and $(1, 0)$. Then,

$$(1 + i, 0)(\alpha, \beta) = (1, 0)$$

implies that

$$(1 + i)\alpha = 1, \quad (1 + i)\beta = 0.$$

If we let $\alpha = a_0 + a_1i + a_2j + a_3k$, then

$$a_0 - a_1 = 1, \quad a_0 + a_1 = 0,$$

which is impossible. That is, there is no element $(\alpha, \beta) \in H_8^{*(2)}$ such that $(1+i,0)(\alpha,\beta) = (1,0)$. Consequently, $H_8^{*(2)}$ is an 8 dimensional nonalternative, noncommutative, nonflexible, nondivision algebra with the identy $(1,0)$. □

From the result of the theorem, we establish that the 8 dimensional algebra $H_8^{*(2)}$ loses the commutative property and flexible property when we extend 4 dimensional commutative, flexible algebra H^* by means of the given construction. On the other hand, H^* and $H_8^{*(2)}$ are both nonalternative algebras.

Now, let

$$e_1 = (1,0), \quad e_2 = (i,0), \quad e_3 = (j,0), \quad e_4 = (k,0)$$
$$e_5 = (0,1), \quad e_6 = (0,i), \quad e_7 = (0,j), \quad e_8 = (0,k)$$

Then the set $\mathbb{B} = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ is a vector space generator of $H_8^{*(2)}$. Also, the multiplication table is given as follows:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_2	e_2	$-e_1$	e_4	e_3	e_6	$-e_5$	e_8	e_7
e_3	e_3	e_4	e_1	$-e_2$	e_7	e_8	e_5	$-e_6$
e_4	e_4	e_3	$-e_2$	$-e_1$	e_8	e_7	$-e_6$	$-e_5$
e_5	e_5	$-e_6$	e_7	$-e_8$	$-e_1$	e_2	$-e_3$	e_4
e_6	e_6	e_5	e_8	$-e_7$	$-e_2$	$-e_1$	$-e_4$	e_3
e_7	e_7	$-e_8$	e_5	e_6	$-e_3$	e_4	$-e_1$	$-e_2$
e_8	e_8	$-e_7$	$-e_6$	e_5	$-e_4$	e_3	e_2	$-e_1$

By the above multiplication table, we obtain the following lemma:

LEMMA 2.6. *$H_8^{*(2)}$ is a $\mathbb{Z}_2$-graded algebra.*

$H_8^{*(2)}$ is not an alternative algebra, but the generating subset $\{e_5, e_6, e_7, e_8\}$ of $H_8^{*(2)}$ satisfies the following multiplication rules:

LEMMA 2.7. *Let $\{e_5, e_6, e_7, e_8\}$ be the generating subset of $H_8^{*(2)}$. Then,*

$$(e_p e_p)e_q = e_p(e_p e_q) \quad or \quad (e_p e_p)e_q = -e_p(e_p e_q), \qquad 5 \le p, q \le 8$$

Now, we want to consider another real matrix representation of elements in $H_8^{*(2)}$ using the conjugate $\alpha^{(2)}$ of α.

THEOREM 2.8. *$H_8^{*(2)}$ is isomorphic to a subspace of $M_{8\times 8}(\mathbb{R})$.*

Proof. Let $(\alpha,\beta) \in H_8^{*(2)}$ be any element. Define a map $\phi^{(2)}$: $H_8^{*(2)} \longrightarrow M_{8\times 8}(\mathbb{R})$ by

$$\phi^{(2)}(\alpha,\beta) = \begin{pmatrix} \phi_1(\alpha) & -\phi_1(\beta)G_2 \\ \phi_1(\beta)G_2 & \phi_1(\alpha) \end{pmatrix},$$

where $\phi_1 : H^* \longrightarrow M_{4\times 4}(\mathbb{R}),\ \alpha = a_0 + a_1 i + a_2 j + a_3 k,$

$$\phi_1(\alpha) = \begin{pmatrix} a_0 & -a_1 & a_2 & -a_3 \\ a_1 & a_0 & -a_3 & -a_2 \\ a_2 & a_3 & a_0 & a_1 \\ a_3 & a_2 & a_1 & a_0 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Then, it is obvious that

$$\phi^{(2)}(a\alpha, a\beta) = a\phi^{(2)}(\alpha,\beta), \quad \phi^{(2)}((\alpha,\beta)+(\gamma,\delta)) = \phi^{(2)}(\alpha,\beta)+\phi^{(2)}(\gamma,\delta)$$

for all elements $(\alpha,\beta),(\gamma,\delta) \in H_8^{*(2)}$ and $a \in \mathbb{R}$.
Also, $\left\{(\alpha,\beta) \in H_8^{*(2)} \,|\, \phi^{(2)}(\alpha,\beta) = O\right\} = \{(0,0)\}$ and so we proved the theorem. □

Throughout this paper, O is the zero matrix of any size for convenience.

THEOREM 2.9. *Let $(\alpha,\beta),(\gamma,\delta) \in H_8^{*(2)}$. Then,*
(1) $\phi^{(2)}(1,0) = I_8$.
(2) $\phi^{(2)}((\alpha,\beta)(\gamma,\delta)) \neq \phi^{(2)}(\alpha,\beta)\phi^{(2)}(\gamma,\delta)$ in general.
(3) $\det(a\phi^{(2)}(\alpha,\beta)) = a^8 \det(\phi^{(2)}(\alpha,\beta))$.

Proof. (1) and (3) are obtained by straightforward computations.
(2) Let $(\alpha,\beta) = (i,0)$ and $(\gamma,\delta) = (j,0)$. Then, $\phi^{(2)}((\alpha,\beta)(\gamma,\delta)) = \begin{pmatrix} \phi_1(k) & O \\ O & \phi_1(k) \end{pmatrix}$. Also, $\phi^{(2)}(\alpha,\beta)\phi^{(2)}(\gamma,\delta) = \begin{pmatrix} \phi_1(i)\phi_1(j) & O \\ O & \phi_1(i)\phi_1(j) \end{pmatrix}$.
But, $\phi_1(i)\phi_1(j) \neq \phi_1(k)$ and so we have (2). □

THEOREM 2.10. *Let $\alpha,\beta \in H^*$. Then,*
(1) If $\phi_1(\alpha)$ is nonsingular if and only if $\phi^{(2)}(\alpha,0)$ is nonsingular.
(2) If $\phi_1(\beta)$ is nonsingular if and only if $\phi^{(2)}(0,\beta)$ is nonsingular.
(3) If $\phi^{(2)}(\alpha,\beta)$ is nonsingular and $\phi_1(\alpha) = 0$, then $\phi_1(\beta)$ is nonsingular.
(4) If $\phi^{(2)}(\alpha,\beta)$ is nonsingular and $\phi_1(\beta) = 0$, then $\phi_1(\alpha)$ is nonsingular.
(5) $tr(\phi^{(2)}(\alpha,\beta)) = 8a_0 = 8Re(\alpha) = 2tr(\phi_1(\alpha))$.

Proof. (1) If $\phi_1(\alpha)$ is nonsingular, then $\phi_1^{-1}(\alpha)$ exists and

$$\phi^{(2)}(\alpha,0)\begin{pmatrix}\phi_1(\alpha)^{-1} & O\\ O & \phi_1(\alpha)^{-1}\end{pmatrix}=I_8.$$

Thus $\phi^{(2)}(\alpha,0)$ is nonsingular.

Conversely, if $\phi^{(2)}(\alpha,0)$ is nonsingular, then

$$\phi^{(2)}(\alpha,0)\begin{pmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{pmatrix}=\begin{pmatrix}\phi_1(\alpha) & O\\ O & \phi_1(\alpha)\end{pmatrix}\begin{pmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{pmatrix}=I_8$$

for some $C_{11},C_{12},C_{21},C_{22}\in M_{4\times4}(\mathbb{R})$. Thus, $\phi_1(\alpha)C_{11}=I_4$ for some $C_{11}\in M_{4\times4}(\mathbb{R})$ and so $\phi_1(\alpha)$ is nonsingular.
(2) If $\phi_1(\beta)$ is nonsingular, then $\phi_1^{-1}(\beta)$ exists and

$$\begin{pmatrix}O & -\phi_1(\beta)G_2\\ \phi_1(\beta)G_2 & O\end{pmatrix}\begin{pmatrix}O & G_2^{-1}\phi_1(\beta)^{-1}\\ -G_2^{-1}\phi_1(\beta)^{-1} & O\end{pmatrix}=I_8.$$

Thus $\phi^{(2)}(0,\beta)$ is nonsingular.

Conversely, if $\phi^{(2)}(0,\beta)$ is nonsingular, then

$$\phi^{(2)}(0,\beta)\begin{pmatrix}D_{11} & D_{12}\\ D_{21} & D_{22}\end{pmatrix}=\begin{pmatrix}O & -\phi_1(\beta)G_2\\ \phi_1(\beta)G_2 & O\end{pmatrix}\begin{pmatrix}D_{11} & D_{12}\\ D_{21} & D_{22}\end{pmatrix}=I_8$$

for some $D_{11},D_{12},D_{21},D_{22}\in M_{4\times4}(\mathbb{R})$. Thus, $\phi_1(\beta)G_2D_{12}=I_4$ for some D_{11} and so $\phi_1(\beta)$ is nonsingular.
(3) If $\phi^{(2)}(\alpha,\beta)$ is nonsingular and $\phi_1(\alpha)=0$, then
$\phi^{(2)}(0,\beta)=\begin{pmatrix}O & -\phi_1(\beta)G_2\\ \phi_1(\beta)G_2 & O\end{pmatrix}$ is nonsingular. Thus, $\phi_1(\beta)$ is nonsingular.
(4) If $\phi^{(2)}(\alpha,\beta)$ is nonsingular and $\phi_1(\beta)=0$, then
$\phi^{(2)}(\alpha,0)=\begin{pmatrix}\phi_1(\alpha) & O\\ O & \phi_1(\alpha)\end{pmatrix}$ is nonsingular. Thus, $\phi_1(\alpha)$ is nonsingular.
(5) Since $\phi^{(2)}(\alpha,\beta)=\begin{pmatrix}\phi_1(\alpha) & -\phi_1(\beta)G_2\\ \phi_1(\beta)G_2 & \phi_1(\alpha)\end{pmatrix}$,
the equalities $tr(\phi^{(2)}(\alpha,\beta))=8a_0=8Re(\alpha)=2tr(\phi_1(\alpha))$ is obvious. □

Just as the algebra $H_8^{*(2)}$, another analogous multiplication will be defined using the conjugate $\alpha^{(3)}$ and an algebra is established as follows:

Definition 2.11. Let $H_8^{*(3)}$ be the algebra defined by the operations on the set K as follows:

$$(\alpha,\beta)+(\gamma,\eta)=(\alpha+\gamma,\beta+\eta)$$

$$a(\alpha,\beta) = (a\alpha, a\beta)$$
$$(\alpha,\beta)(\gamma,\eta) = (\alpha\gamma - \beta\eta^{(3)}, \alpha\eta + \beta\gamma^{(3)})$$

for all elements $(\alpha,\beta),(\gamma,\eta) \in K$ and $a \in \mathbb{R}$.

THEOREM 2.12. *$H_8^{*(3)}$ is an 8 dimensional nonalternative, nonflexible, noncommutative, nondivision algebra with identity.*

Proof. The proof is similar to case of the algebra $H_8^{*(2)}$. We only prove the nonalternative property and nonflexible property. Note that

$$\{(i,j)(i,j)\}\,(i,i) = (-2j, 2j) \neq (-2i, -2i) = (i,j)\,\{(i,j)(i,i)\}$$

and so $H_8^{*(3)}$ is not an alternative algebra.

Moreover, $H_8^{*(3)}$ is a nonflexible algebra since

$$\{(i,j)(j,i)\}\,(i,j) = (-2j, -2i) \neq (2j, 2i) = (i,j)\,\{(j,i)(i,j)\}\,.$$

Consequently, $H_8^{*(2)}$ is an 8 dimensional nonalternative, noncommutative, nonflexible, nondivision algebra with identy. □

Now, we let

$$e_1 = (1,0), \quad e_2 = (i,0), \quad e_3 = (j,0), \quad e_4 = (k,0)$$
$$e_5 = (0,1), \quad e_6 = (0,i), \quad e_7 = (0,j), \quad e_8 = (0,k)$$

Then the set $\mathbb{B} = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ is a vector space generator of $H_8^{*(3)}$. Also, the multiplication table is given as follows:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_2	e_2	$-e_1$	e_4	e_3	e_6	$-e_5$	e_8	e_7
e_3	e_3	e_4	e_1	$-e_2$	e_7	e_8	e_5	$-e_6$
e_4	e_4	e_3	$-e_2$	$-e_1$	e_8	e_7	$-e_6$	$-e_5$
e_5	e_5	e_6	$-e_7$	$-e_8$	$-e_1$	$-e_2$	e_3	e_4
e_6	e_6	$-e_5$	$-e_8$	$-e_7$	$-e_2$	e_1	e_4	e_3
e_7	e_7	e_8	$-e_5$	e_6	$-e_3$	$-e_4$	e_1	$-e_2$
e_8	e_8	e_7	e_6	e_5	$-e_4$	$-e_3$	$-e_2$	$-e_1$

LEMMA 2.13. *$H_8^{*(3)}$ is a $\mathbb{Z}_2$ graded algebra.*

The following result is straightforward.

LEMMA 2.14. *The generating set $\{e_5, e_6, e_7, e_8\}$ of $H_8^{*(3)}$ satisfies the following multiplication rules:*

$$(e_p e_p)e_q = e_p(e_p e_q) \quad or \quad (e_p e_p)e_q = -e_p(e_p e_q), \qquad 5 \leq p, q \leq 8$$

Now, we want to consider a real matrix representation of elements in H_8^*. Let $(\alpha,\beta) \in H_8^{*(3)}$ and let $\alpha = a_0 + a_1 i + a_2 j + a_3 k$ and $\beta = b_0 + b_1 i + b_2 j + b_3 k$. Then, with respect to the vector space basis $\mathbb{B} = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$, we obtain a matrix representation

$$\phi^{(3)}(\alpha,\beta) = \begin{pmatrix} a_0 & -a_1 & a_2 & -a_3 & -b_0 & b_1 & b_2 & -b_3 \\ a_1 & a_0 & -a_3 & -a_2 & -b_1 & -b_0 & -b_3 & -b_2 \\ a_2 & a_3 & a_0 & a_1 & -b_2 & -b_3 & b_0 & b_1 \\ a_3 & a_2 & a_1 & a_0 & -b_3 & -b_2 & b_1 & b_0 \\ b_0 & -b_1 & -b_2 & b_3 & a_0 & -a_1 & a_2 & -a_3 \\ b_1 & b_0 & b_3 & b_2 & a_1 & a_0 & -a_3 & -a_2 \\ b_2 & b_3 & -b_0 & -b_1 & a_2 & a_3 & a_0 & a_1 \\ b_3 & b_2 & -b_1 & -b_0 & a_3 & a_2 & a_1 & a_0 \end{pmatrix}$$

Moreover, if we define a map $\phi^{(3)} : H_8^{*(3)} \longrightarrow M_{8\times 8}(\mathbb{R})$ by

$$\phi^{(3)}(\alpha,\beta) = \begin{pmatrix} \phi_1(\alpha) & -\phi_1(\beta)G_3 \\ \phi_1(\beta)G_3 & \phi_1(\alpha) \end{pmatrix},$$

where $\phi_1 : H^* \longrightarrow M_{4\times 4}(\mathbb{R})$,

$$\phi_1(\alpha) = \begin{pmatrix} a_0 & -a_1 & a_2 & -a_3 \\ a_1 & a_0 & -a_3 & -a_2 \\ a_2 & a_3 & a_0 & a_1 \\ a_3 & a_2 & a_1 & a_0 \end{pmatrix}, \qquad G_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

then we have the next theorem.

THEOREM 2.15. *$H_8^{*(3)}$ is isomorphic to a subspace of $M_{8\times 8}(\mathbb{R})$.*

The following properties can be proved obviously:

THEOREM 2.16. *Let $(\alpha,\beta),(\gamma,\delta) \in H_8^{*(3)}$. Then,*
(1) $\phi^{(3)}(1,0) = I_8$.
(2) $\phi^{(3)}((\alpha,\beta)(\gamma,\delta)) \neq \phi^{(3)}(\alpha,\beta)\phi^{(3)}(\gamma,\delta)$ in general.
(3) $\det(a\phi^{(3)}(\alpha,\beta)) = a^8 \det(\phi^{(3)}(\alpha,\beta))$.

THEOREM 2.17. *Let $\alpha,\beta \in H^*$. Then,*
(1) If $\phi_1(\alpha)$ is nonsingular if and only if $\phi^{(3)}(\alpha,0)$ is nonsingular.
(2) If $\phi_1(\beta)$ is nonsingular if and only if $\phi^{(3)}(0,\beta)$ is nonsingular.
(3) If $\phi^{(3)}(\alpha,\beta)$ is nonsingular and $\phi_1(\alpha) = 0$, then $\phi_1(\beta)$ is nonsingular.
(4) If $\phi^{(3)}(\alpha,\beta)$ is nonsingular and $\phi_1(\beta) = 0$, then $\phi_1(\alpha)$ is nonsingular.
(5) $tr(\phi^{(3)}(\alpha,\beta)) = 8a_0 = 8Re(\alpha) = 2tr(\phi_1(\alpha))$.

Proof. We only prove (2) and (3).

(2) If $\phi_1(\beta)$ is nonsingular, then $\phi_1^{-1}(\beta)$ exists and

$$\begin{pmatrix} O & -\phi_1(\beta)G_3 \\ \phi_1(\beta)G_3 & O \end{pmatrix} \begin{pmatrix} O & G_3^{-1}\phi_1(\beta)^{-1} \\ -G_3^{-1}\phi_1(\beta)^{-1} & O \end{pmatrix} = I_8.$$

Thus $\phi^{(3)}(0,\beta)$ is nonsingular.

Conversely, if $\phi^{(3)}(0,\beta)$ is nonsingular, then

$$\phi^{(3)}(0,\beta) \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} = \begin{pmatrix} O & -\phi_1(\beta)G_3 \\ \phi_1(\beta)G_3 & O \end{pmatrix} \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} = I_8$$

for some $E_{11}, E_{12}, E_{21}, E_{22} \in M_{4\times 4}(\mathbb{R})$. Thus, $\phi_1(\beta)G_3E_{12} = I_4$ for some E_{12} and so $\phi_1(\beta)$ is nonsingular.

(3) If $\phi^{(3)}(\alpha,\beta)$ is nonsingular and $\phi_1(\alpha) = 0$, then
$\phi^{(3)}(0,\beta) = \begin{pmatrix} O & -\phi_1(\beta)G_3 \\ \phi_1(\beta)G_3 & O \end{pmatrix}$ is nonsingular. Thus, $\phi_1(\beta)$ is nonsingular. □

3. Properties of Matrices in $M_{n\times n}(H_8^{*(m)})$

Let $\psi^{(m)} : M_{n\times n}(H_8^{*(m)}) \longrightarrow M_{8n\times 8n}(\mathbb{R})$, $m = 2, 3$ be the map defined by $\psi^{(m)}(A) = C$, where $A = (A_{st})_{n\times n}$, $A_{st} = (\alpha_{st}, \beta_{st})$, $C = (C_{st})_{8n\times 8n}$, and

$$C_{st} = \phi^{(m)}(A_{st}) = \begin{pmatrix} \phi_1(\alpha_{st}) & -\phi_1(\beta_{st})G_m \\ \phi_1(\beta_{st})G_m & \phi_1(\alpha_{st}) \end{pmatrix}.$$

Definition 3.1. Let $A \in M_{n\times n}(H_8^{*(m)})$, $m = 2, 3$. Then, the $8n\times 8n$ matrix $\psi^{(m)}(A)$ is called the adjoint matrix of A with respect to the m-th conjugate.

Theorem 3.2. *Let $A, B \in M_{n\times n}(H_8^{*(m)})$, $m = 2, 3$. Then,*

(1) $\psi^{(m)}(aA) = a\psi^{(m)}(A)$ for all $a \in \mathbb{R}$.

(2) $\psi^{(m)}(A + B) = \psi^{(m)}(A) + \psi^{(m)}(B)$.

(3) $\psi^{(m)}(AB) \neq \psi^{(m)}(A)\psi^{(m)}(B)$ in general.

Proof. Let $A = (A_{st})_{n\times n}$, $A_{st} = (\alpha_{st}, \beta_{st})$, $B = (B_{st})_{n\times n}$, $B_{st} = (\gamma_{st}, \delta_{st})$ for some $(\alpha_{st}, \beta_{st}), (\gamma_{st}, \delta_{st}) \in H_8^*$. Then,

(1) The $(s,t) - th$ 8×8 block matrix of $\psi^{(m)}(aA)$ is

$$\phi^{(m)}(aA_{st}) = \phi^{(m)}(a\alpha_{st}, a\beta_{st}) = a\phi^{(m)}(\alpha_{st}, \beta_{st}) = a\phi^{(m)}(A_{st}).$$

which is the $(s,t)-th$ 8×8 block matrix of $a\psi^{(m)}(A)$. Thus, $\psi^{(m)}(aA) = a\psi^{(m)}(A)$.

(2) The $(s,t)-th$ 8×8 block matrix of $\psi^{(m)}(A+B)$ is

$$\begin{array}{rcl} \phi^{(m)}(A_{st}+B_{st}) & = & \phi^{(m)}(\alpha_{st}+\gamma_{st}, \beta_{st}+\delta_{st}) \\ & = & \begin{pmatrix} \phi_1(\alpha_{st}+\gamma_{st}) & -\phi_1(\beta_{st}+\delta_{st})G_m \\ \phi_1(\beta_{st}+\delta_{st})G_m & \phi_1(\alpha_{st}+\gamma_{st}) \end{pmatrix} \\ & = & \phi^{(m)}(A_{st})+\phi^{(m)}(B_{st}) \end{array}$$

which is the $(s,t)-th$ 8×8 block matrix of $\psi^{(m)}(A)+\psi^{(m)}(B)$.

(3) We will prove for $m=2$. Let $A=(i,0)I_n$ and $B=(j,0)I_n$. Then, $AB=(k,0)I_n$ and

$$\begin{array}{rcl} \psi^{(2)}(A) & = & diag(\phi^{(2)}(i,0), \phi^{(2)}(i,0), \cdots, \phi^{(2)}(i,0)) \\ & = & diag\left(\begin{pmatrix} \phi_1(i) & O \\ O & \phi_1(i) \end{pmatrix}, \cdots, \begin{pmatrix} \phi_1(i) & O \\ O & \phi_1(i) \end{pmatrix}\right) \\ \psi^{(2)}(B) & = & diag(\phi^{(2)}(j,0), \phi^{(2)}(j,0), \cdots, \phi^{(2)}(j,0)) \\ & = & diag\left(\begin{pmatrix} \phi_1(j) & O \\ O & \phi_1(j) \end{pmatrix}, \cdots, \begin{pmatrix} \phi_1(j) & O \\ O & \phi_1(j) \end{pmatrix}\right) \\ \psi^{(2)}(AB) & = & diag(\phi^{(2)}(k,0), \phi^{(2)}(k,0), \cdots, \phi^{(2)}(k,0)) \\ & = & diag\left(\begin{pmatrix} \phi_1(k) & O \\ O & \phi_1(k) \end{pmatrix}, \cdots, \begin{pmatrix} \phi_1(k) & O \\ O & \phi_1(k) \end{pmatrix}\right) \end{array}$$

Since $\phi_1(i)\phi_1(j) \neq \phi_1(k)$, we have $\psi^{(2)}(AB) \neq \psi^{(2)}(A)\psi^{(2)}(B)$. □

DEFINITION 3.3. Let $A = ((\alpha_{st}, \beta_{st}))_{n\times n} \in M_{n\times n}(H_8^{*(m)})$, $m=2,3$. Then, $A^{(m)} = ((\alpha_{st}^{(m)}, \beta_{st}^{(m)}))_{n\times n}$ is called the m-th conjugate of A.

THEOREM 3.4. *Let* $A, B \in M_{n\times n}(H_8^{*(m)})$, $m=2,3$. *Then,* $(AB)^{(m)} = A^{(m)}B^{(m)}$.

Proof. Let $A=(A_{st})_{n\times n}$ and $B=(B_{st})_{n\times n}$, where $A_{st}=(\alpha_{st},\beta_{st})$ and $B_{st}=(\gamma_{st},\delta_{st})$. Then, pq entry of AB is

$$\sum_{s=1}^{n} A_{ps}B_{sq} = \sum_{s=1}^{n}(\alpha_{ps},\beta_{ps})(\gamma_{sq},\delta_{sq}) = \sum_{s=1}^{n}(\alpha_{ps}\gamma_{sq}-\beta_{ps}\delta_{sq}^{(m)}, \alpha_{ps}\delta_{sq}+\beta_{ps}\gamma_{sq}^{(m)}).$$

Thus, pq entry of $(AB)^{(m)}$ is

$$\sum_{s=1}^{n}(\alpha_{ps}^{(m)}\gamma_{sq}^{(m)} - \beta_{ps}^{(m)}\delta_{sq}, \alpha_{ps}^{(m)}\delta_{sq}^{(m)} + \beta_{ps}^{(m)}\gamma_{sq}).$$

which is the pq entry of $A^{(m)}B^{(m)}$. Thus, $(AB)^{(m)} = A^{(m)}B^{(m)}$. □

THEOREM 3.5. *Let* $A, B \in M_{n\times n}(H_8^{*(m)})$, $m = 2, 3$. *Then,*

(1) $\psi^{(m)}(aA^{(m)}) = a\psi^{(m)}(A^{(m)})$ *for all* $a \in \mathbb{R}$.

(2) $\psi^{(m)}((A + B)^{(m)}) = \psi^{(m)}(A^{(m)}) + \psi^{(m)}(B^{(m)})$.

(3) $\psi^{(m)}((AB)^{(m)}) \neq \psi^{(m)}(A^{(m)})\psi^{(m)}(B^{(m)})$ *in general.*

Proof. (1) and (2) are obvious and we will prove only (3).
(3) We will prove only for $m = 3$ and the case of $m = 2$ can be proved similarly. Let $A = (i, 0)I_n$ and $B = (j, 0)I_n$. Then, $AB = (k, 0)I_n$, $A^{(3)} = (i, 0)I_n$, $B^{(3)} = (-j, 0)I_n$, $(AB)^{(3)} = (-k, 0)I_n$. Also,

$$\begin{aligned}\psi^{(3)}(A^{(3)}) &= diag(\phi^{(3)}(i,0), \phi^{(3)}(i,0), \cdots, \phi^{(3)}(i,0)) \\ &= diag\left(\begin{pmatrix} \phi_1(i) & O \\ O & \phi_1(i)\end{pmatrix}, \cdots, \begin{pmatrix} \phi_1(i) & O \\ O & \phi_1(i)\end{pmatrix}\right)\end{aligned}$$

$$\begin{aligned}\psi^{(3)}(B^{(3)}) &= diag(\phi^{(3)}(-j,0), \phi^{(3)}(-j,0), \cdots, \phi^{(3)}(-j,0)) \\ &= diag\left(\begin{pmatrix} \phi_1(-j) & O \\ O & \phi_1(-j)\end{pmatrix}, \cdots, \begin{pmatrix} \phi_1(-j) & O \\ O & \phi_1(-j)\end{pmatrix}\right)\end{aligned}$$

$$\begin{aligned}\psi^{(3)}((AB)^{(3)}) &= diag(\phi^{(3)}(-k,0), \phi^{(3)}(-k,0), \cdots, \phi^{(3)}(-k,0)) \\ &= diag\left(\begin{pmatrix} \phi_1(-k) & O \\ O & \phi_1(-k)\end{pmatrix}, \cdots, \begin{pmatrix} \phi_1(-k) & O \\ O & \phi_1(-k)\end{pmatrix}\right)\end{aligned}$$

Therefore, $\psi^{(3)}((AB)^{(3)}) \neq \psi^{(3)}(A^{(3)})\psi^{(3)}(B^{(3)})$. □

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